Chapter 4

The Three-Moment Equations-I

Instructional Objectives

After reading this chapter the student will be able to

1. Derive three-moment equations for a continuous beam with unyielding supports.
2. Write compatibility equations of a continuous beam in terms of three moments.
3. Compute reactions in statically indeterminate beams using three-moment equations.
4. Analyse continuous beams having different moments of inertia in different spans using three-moment equations.

Introduction

Beams that have more than one span are defined as continuous beams. Continuous beams are very common in bridge and building structures. Hence, one needs to analyze continuous beams subjected to transverse loads and support settlements quite often in design. When beam is continuous over many supports and moment of inertia of different spans is different, the force method of analysis becomes quite cumbersome if vertical components of reactions are taken as redundant reactions. However, the force method of analysis could be further simplified for this particular case (continuous beam) by choosing the unknown bending moments at the supports as unknowns. One compatibility equation is written at each intermediate support of a continuous beam in terms of the loads on the adjacent span and bending moment at left, center (the support where the compatibility equation is written) and rigid supports. Two consecutive spans of the continuous beam are considered at one time. Since the compatibility equation is written in terms of three moments, it is known as the equation of three moments. In this manner, each span is treated individually as a simply supported beam with external loads and two end support moments. For each intermediate support, one compatibility equation is written in terms of three moments. Thus, we get as many equations as there are unknowns. Each equation will have only three unknowns. It may be noted that, Clapeyron first proposed this method in 1857. In this lesson, three moment equations are derived for unyielding supports and in the next lesson the three moment equations are modified to consider support moments.

Three-moment equation

A continuous beam is shown in Fig.12.1a. Since, three moment equation relates moments at three successive supports to applied loading on adjacent spans, consider two adjacent spans of a continuous beam as shown in Fig.12.1b. M_L, M_C and M_R respectively denote support moments at left, center and right supports. The moments are taken to be positive when they cause tension at bottom fibers. The moment of inertia is taken to be different for different spans. In the present case I_L and I_R denote respectively moment of inertia of; left and right support and l_L and l_R are the left and right span respectively. It is assumed that supports are unyielding. The yielding of supports could be easily incorporated in three-moment equation, which will be discussed in the next lesson. Now it is required to derive a relation between M_L, M_C and M_R. This relationship is derived from the fact that the tangent to the elastic curve at C is horizontal. In other words the joint C may be considered rigid. Thus, the compatibility equation is written as,

$$\theta_{CL} + \theta_{CR} = 0 \tag{12.1}$$

The rotation left of the support C, θ_{CL} and rotation right of the support C, θ_{CR} may be calculated from moment area method. Now,

$$\theta_{CL} = \frac{\text{Deflection of L from tangent drawn at C(LL}'\text{)}}{l_L}$$

$$= \frac{\text{Moment of } {}^{\text{M}}\!/\!{}_{\text{EI}} \text{ diagram between C and L about L}}{l_L}$$

$$= \frac{1}{l_L}\left[\left(\frac{A_L x_L}{EI_L}\right) + \frac{1}{2}\left(\frac{M_L}{EI_L}\right) l_L \frac{1}{3} l_L + \frac{1}{2}\left(\frac{M_C}{EI_L}\right) l_L \frac{2}{3} l_L\right]$$

$$\theta_{CL} = \frac{A_L \bar{x}_L}{EI_L l_L} + \frac{M_L l_L}{6EI_L} + \frac{M_C l_L}{3EI_L} \tag{12.2}$$

Note that the actual moment diagram on span LC is broken into two parts (1) due to loads applied on span LC when it is considered as a simply supported beam and, (2) due to support moments. In the above equation A_L and A_R denote respectively area of the bending moment diagrams due to applied loads on left and right supports. x_L and x_R denote their respective C.G.(center of gravity) distances from the left and right support respectively. Similarly,

$$\theta_{CR} = \frac{\text{deflection of R from tangent drawn at C (RR')}}{l_R}$$

$$= \frac{\text{Moment of } {}^{\text{M}}\!/\!{}_{\text{EI}} \text{ diagram between C and R about R}}{l_R}$$

$$\theta_{CR} = \frac{A_R \bar{x}_R}{EI_R l_R} + \frac{M_R l_R}{6EI_R} + \frac{M_C l_R}{3EI_R} \quad (12.3)$$

Substituting the values of θ_{CL} and θ_{CR} in the compatibility equation (12.1),

$$\frac{A_L \bar{x}_L}{EI_L l_L} + \frac{M_L l_L}{6EI_L} + \frac{M_C l_L}{3EI_L} + \frac{A_R \bar{x}_R}{EI_R l_R} + \frac{M_R l_R}{6EI_R} + \frac{M_C l_R}{3EI_R} = 0 \quad (12.4)$$

which could be simplified to,

$$M_L \left(\frac{l_L}{I_L}\right) + 2M_C \left(\frac{l_L}{I_L} + \frac{l_R}{I_R}\right) + M_R \left(\frac{l_R}{I_R}\right) = -\frac{6A_R \bar{x}_R}{I_R l_R} - \frac{6A_L \bar{x}_L}{I_L l_L} \quad (12.5)$$

The above equation (12.5) is known as the three-moment equation. It relates three support moments M_L, M_C and M_R with the applied loading on two adjacent spans. If in a span there are more than one type of loading (for example, uniformly distributed load and a concentrated load) then it is simpler to calculate moment diagram separately for each of loading and then to obtain moment diagram.

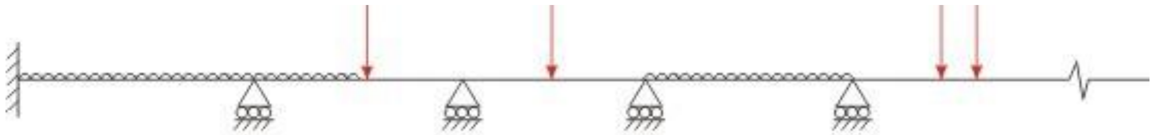
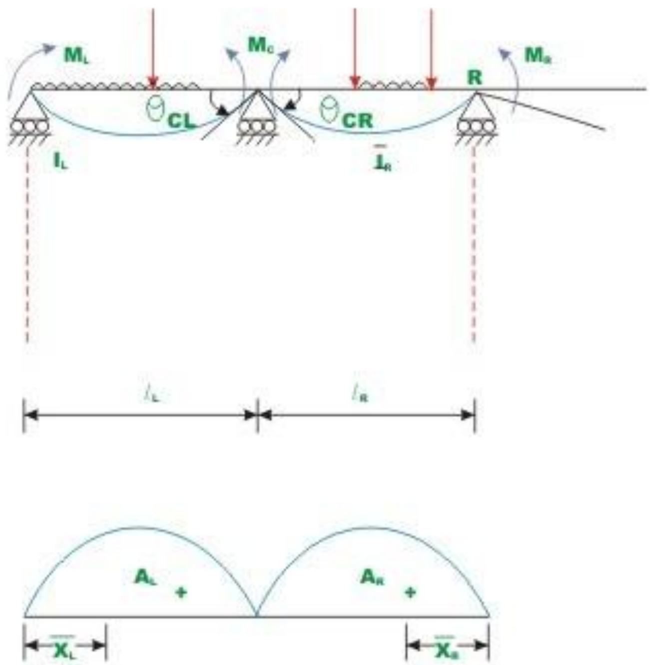
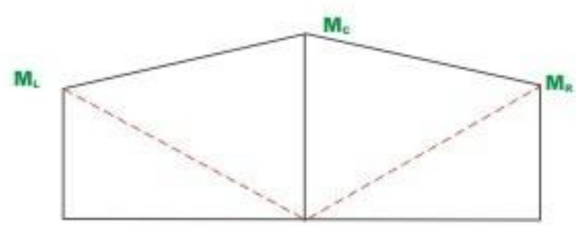


Fig. 12.1 (a) Continuous beam.



Bending moment diagram due to applied loading.



Bending moment diagram (B.M.D) due to support moments.

Fig. 12. 1(b) Two adjacent spans of a continuous beam.

Alternate derivation

The above three moment equations may also be derived by direct application of force method as follows. Now choose M_L, M_C and the M_R, the three support moments at left, centre and right supports respectively as the redundant moments. The primary determinate structure is obtained by releasing the constraint corresponding to redundant moments. In this particular case, inserting hinges at L, C and R, the primary structure is obtained as below (see Fig. 12.2)

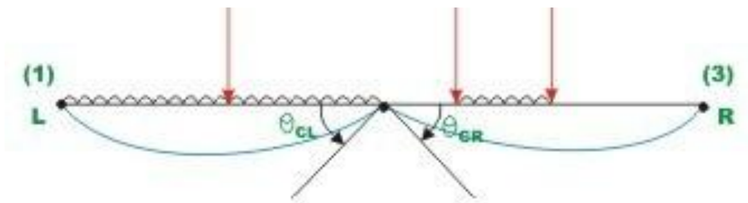


Fig. 12.2. Primary structure

Let displacement (in the primary case rotations) corresponding to rotation M_C be θ_L, which is the sum of rotations θ_{CL} and θ_{CR}. Thus,

$$\theta_L = \theta_{CL} + \theta_{CR} \tag{12.6}$$

It is observed that the rotations θ_{CL} and θ_{CR} are caused due to only applied loading as shown in Fig.12.2.This can be easily evaluated by moment area method as shown previously.

$$\theta_L = \frac{A_L \bar{x}_L}{EI_L l_L} + \frac{A_R x_R}{EI_R l_R} \tag{12.7}$$

In the next step, apply unit value of redundant moments at L, C and R and calculate rotation at C (i.e. flexibility coefficients).

$$a_{21} = \frac{l_L}{6EI_L}$$
$$a_{22} = \frac{l_L}{3EI_L} + \frac{l_R}{3EI_R} \tag{12.8}$$
$$a_{23} = \frac{l_R}{6EI_R}$$

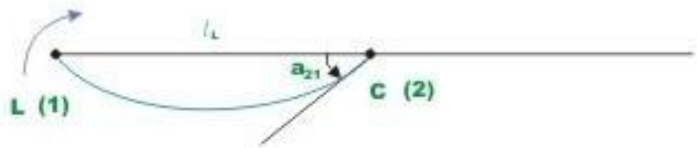


Fig. 12.3 (a) Unit redundant force applied at L (1)

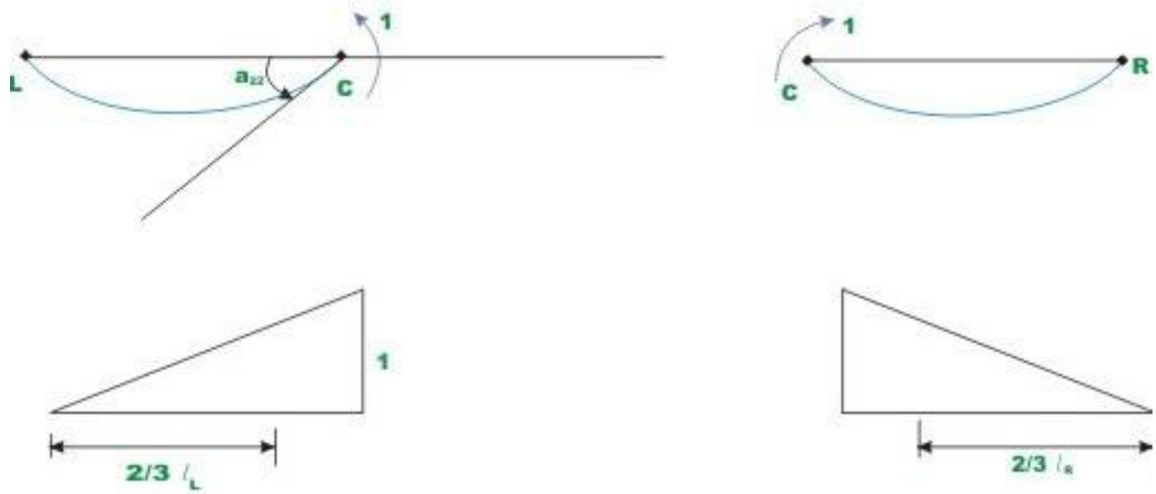


Fig. 12.3 (b) Unit redundant force applied at c.

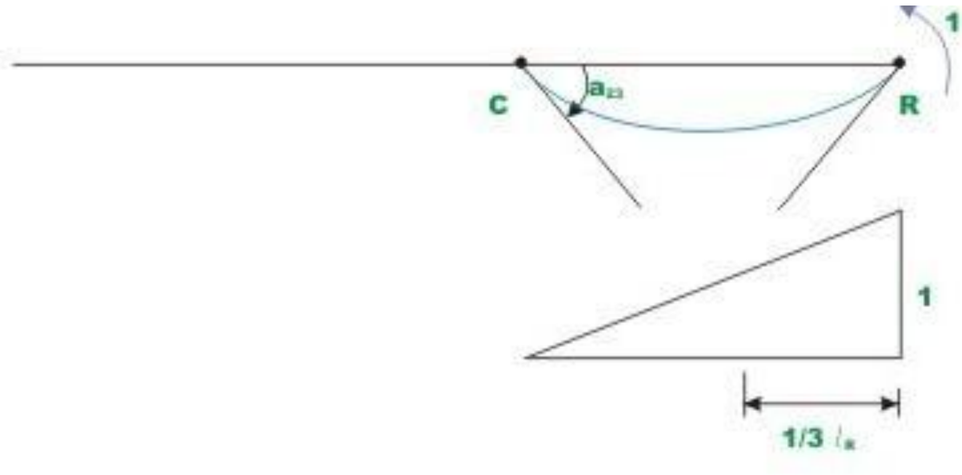


Fig. 12.3 (c) Unit moment applied at R

In the actual structure the relative rotation of both sides is zero. In other words the compatibility equation is written as,

$$\theta_L + a_{21} M_L + a_{22} M_C + a_{23} M_R = 0 \qquad (12.9)$$

Substituting the values of flexibility coefficients and θ_L in the above equation,

$$\frac{A_R \bar{x}_R}{EI_R l_R} + \frac{A_L \bar{x}_L}{EI_L l_L} + M_L\left(\frac{l_L}{6EI_L}\right) + M_C\left(\frac{l_L}{3EI_L} + \frac{l_R}{3EI_R}\right) + M_R\left(\frac{l_R}{6EI_R}\right) = 0$$

Or,

$$M_L\left(\frac{l_L}{I_L}\right) + 2M_C\left(\frac{l_L}{I_L} + \frac{l_R}{I_R}\right) + M_R\left(\frac{l_R}{I_R}\right) = -\frac{6A_R \bar{x}_R}{I_R l_R} - \frac{6A_L \bar{x}_L}{I_L l_L} \qquad (12.10)$$

when moment of inertia remains constant i.e. $I_R = I_L = I$,the above equation simplifies to,

$$M_L(l_L) + 2M_C\{l_L + l_R\} + M_R(l_R) = -\frac{6A_R \bar{x}_R}{l_R} - \frac{6A_L \bar{x}_L}{l_L} \qquad (12.11)$$

Example

A continuous beam *ABCD* is carrying a uniformly distributed load of 1 kN/m over span ABC in addition to concentrated loads as shown in Fig.12.4a. Calculate support reactions. Also, draw bending moment and shear force diagram. Assume EI to be constant for all members.

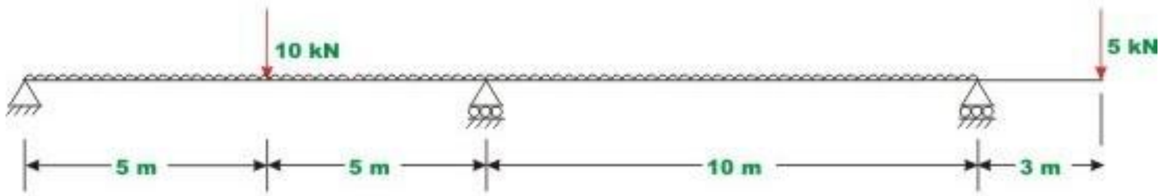


Fig. 12.4 (a) Continuous beam of Example 12 .1

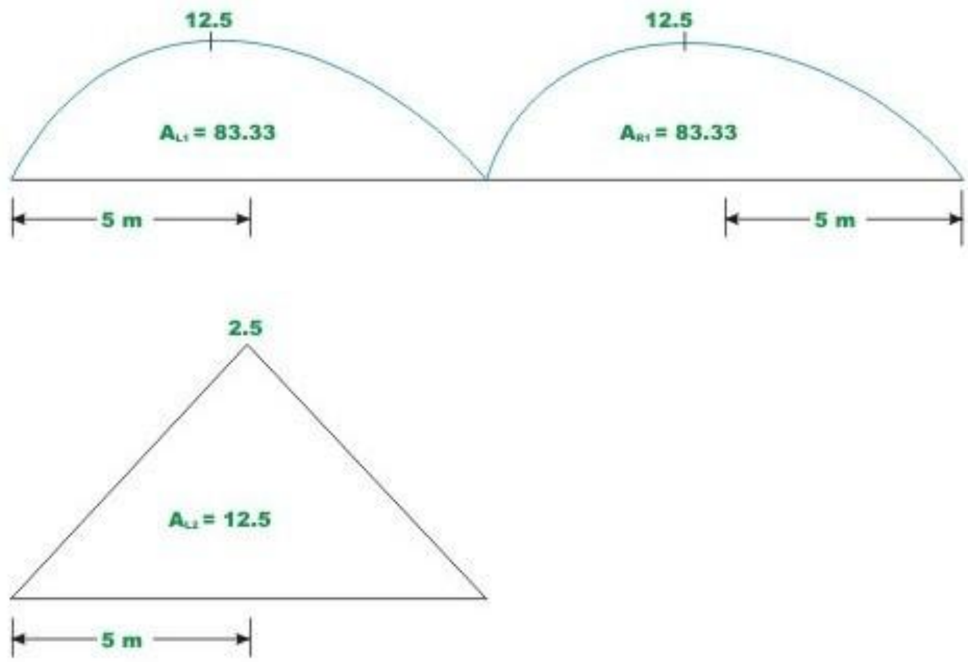


Fig. 12.4 (b) Bending moment diagram due to applied loading

From inspection, it is assumed that the support moments at A is zero and support moment at C,

$M_C = 15$ kN.m (negative because it causes compression at bottom at C)

Hence, only one redundant moment M_B needs to be evaluated. Applying three-moment equation to span ABC,

$$2M_C\{10+10\}+M_C(10)=-\frac{6A_R\bar{x}_R}{l_R}-\frac{6A_L\bar{x}_L}{l_L} \qquad (1)$$

The bending moment diagrams for each span due to applied uniformly distributed and concentrated load are shown in Fig.12.4b.

Equation (1) may be written as,

$$40M_B - 150 = -\frac{6 \times 83.33 \times 5}{10} - \frac{6 \times 125 \times 5}{10} - \frac{6 \times 83.33 \times 5}{10}$$

Thus,

$$M_B = -18.125 \quad \text{kN.m}$$

After determining the redundant moment, the reactions are evaluated by equations of static equilibrium. The reactions are shown in Fig.12.4c along with the external load and support bending moment.

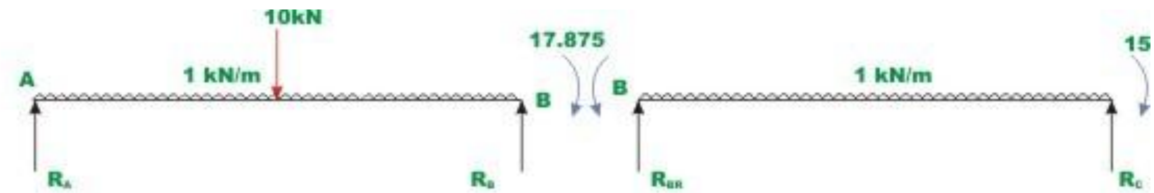
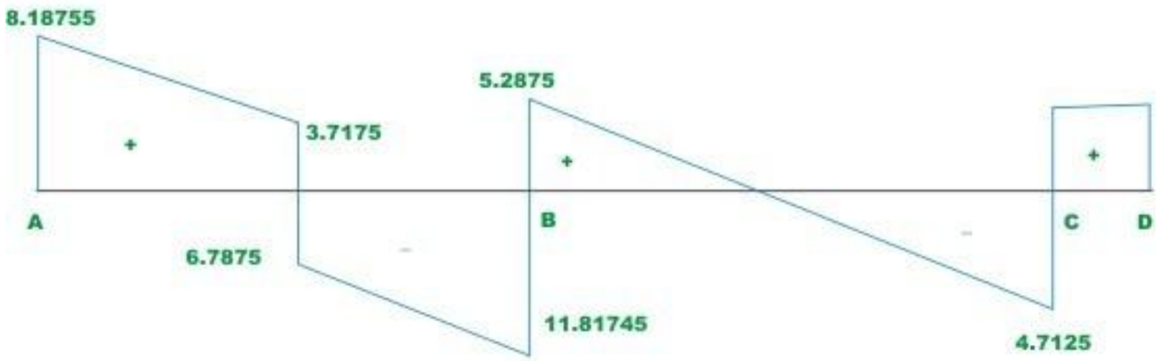
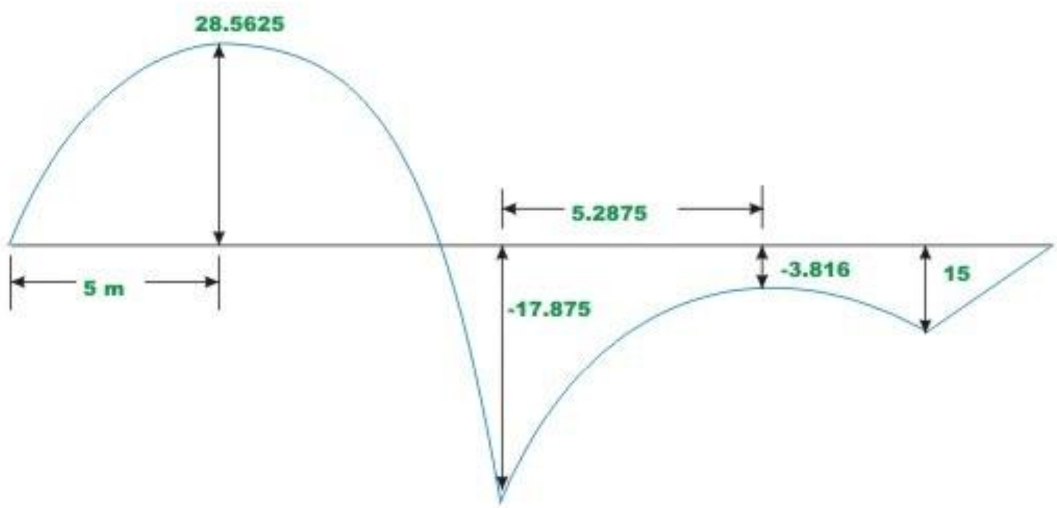


Fig. 12.4 (c) Free - body diagram of two members



Shear force diagram (S.F.D)



Bending moment diagram (B.M.D)

Fig. 12.4(d). SHEARE FORCE & BENDING MOMENT DIAGRAM.

In span AB, R_A can be calculated by the condition that $\sum M_B = 0$. Thus,

$$R_A \times 10 - 10 \times 5 - 10 \times 5 + 18.125 = 0$$

$$R_A = 8.1875 \text{ kN} \qquad (\uparrow)$$

$$R_{BL} = 11.8125 \text{ kN} \qquad (\uparrow)$$

Similarly from span BC,

$$R_C = 4.7125 \text{ kN} \qquad (\uparrow)$$

$$R_{BR} = 5.3125 \text{ kN} \qquad (\uparrow)$$

The shear force and bending moment diagrams are shown in Fig.12.4d.

Example

A continuous beam ABC is carrying uniformly distributed load of 2 kN/m as shown in Fig.12.5a.The moment of inertia of span AB is twice that of span BC. Evaluate reactions and draw bending moment and shear force diagrams.

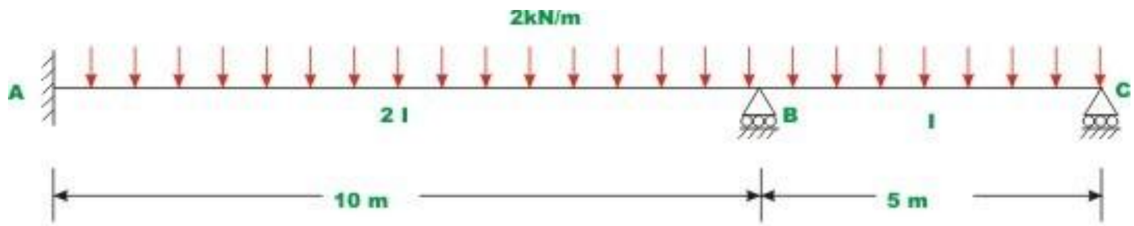


Fig. 12.5 (a) Example 12.2

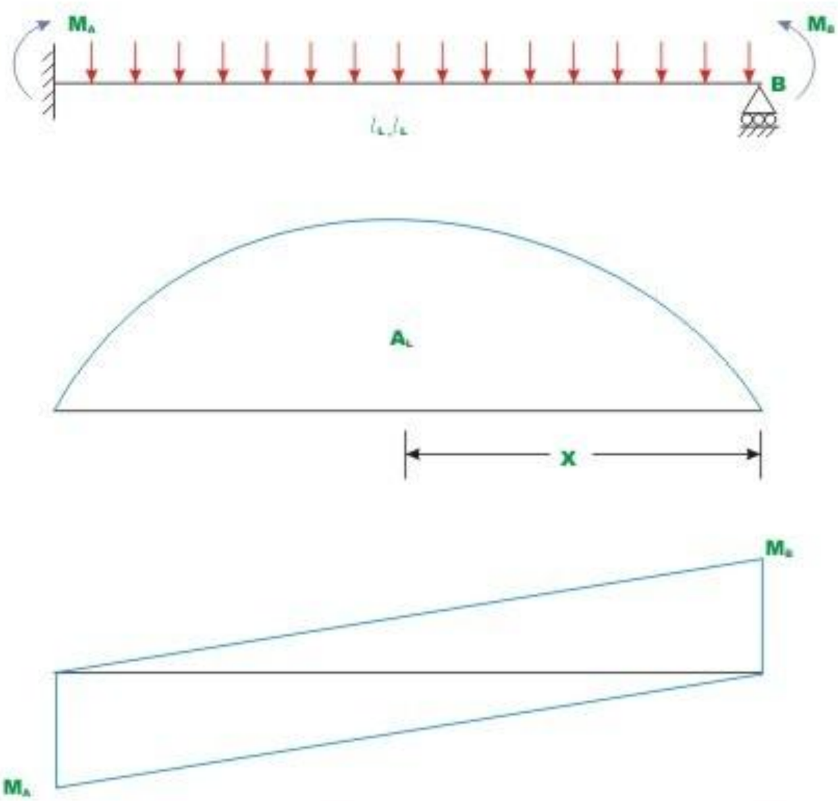


Fig. 12.5(b) Free body diagram of span AB

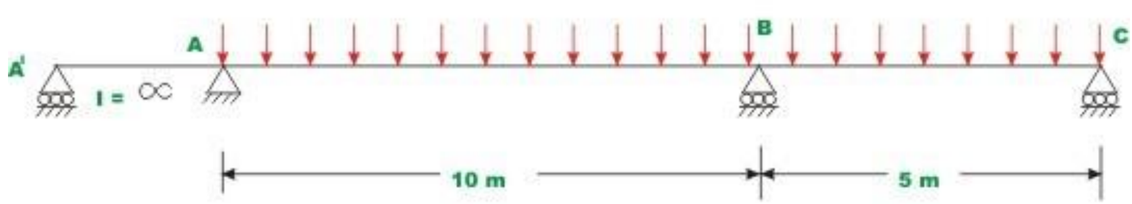


Fig. 12.5(c) Continuous beam within imaginary span AA'

By inspection it is seen that the moment at support C is zero. The support moment at A and B needs to be evaluated .For moment at B, the compatibility equation is written by noting that the tangent to the elastic curve at B is horizontal .The compatibility condition corresponding to redundant moment at A is written as follows. Consider span AB as shown in Fig.12.5b.

The slope at A , θ_A may be calculated from moment-area method. Thus,

$$\theta_A = \frac{M_B l_L}{6EI_L} + \frac{M_A l_L}{3EI_L} + \frac{A(\bar{x}_L)_R}{EIl_L} \qquad (1)$$

Now, compatibility equation is,

$$\theta_A = 0 \qquad (2)$$

It is observed that the tangent to elastic curve at A remains horizontal. This can also be achieved as follows. Assume an imaginary span AA' of length L' left of support A having a very high moment of inertia (see Fig. 12.5c). As the imaginary span has very high moment of inertia, it does not yield any imaginary span has very high moment of inertia it does not yield any M/EI diagram and hence no elastic curve. Hence, the tangent at A to elastic curve remains horizontal.

Now, consider the span $A'AB$, applying three-moment equation to support A ,

$$2M_A\left(\frac{L'}{\infty} + \frac{10}{2I}\right) + M_B\left(\frac{10}{2I}\right) = -\frac{6A_R x_R}{2I(10)} \qquad (3)$$

The above equation is the same as the equation (2). The simply supported bending moment diagram is shown in Fig.12.5d.

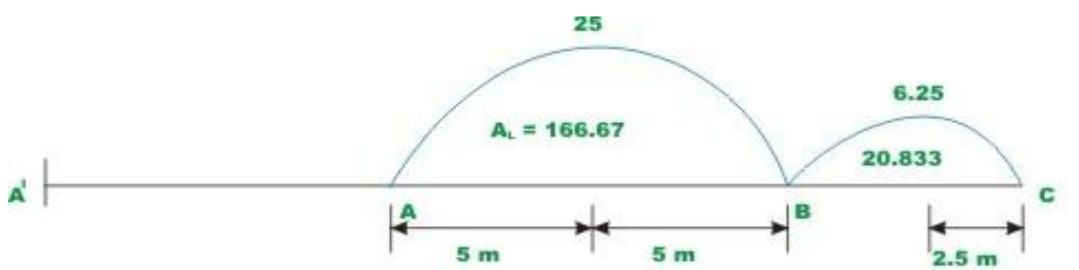


Fig. 12.5 (d) Bending moment diagram due to applied loading

Thus, equation (3) may be written as,

$$20M_A + M_B(10) = -\frac{6 \times (166.67) \times 5}{10}$$

$$20M_A + 10M_B = -500 \qquad (4)$$

Now, consider span ABC, writing three moment equation for support B,

$$M_A\left(\frac{10}{2I}\right) + 2M_B\left(\frac{10}{2I} + \frac{5}{I}\right) = -\frac{6 \times 166.67 \times 5}{2I \times (10)} - \frac{6 \times 20.837 \times 2.5}{I \times (5)}$$

$$5M_A + 20M_B = -250 - 62.5 \qquad (5)$$

$$= -312.5$$

Solving equation (4) and (5),

$$M_B = -6.25 \text{ kN.m}$$

$$M_A = -37.5 \text{ kN.m}$$

The remaining reactions are calculated by equilibrium equations (see Fig.12.5e)

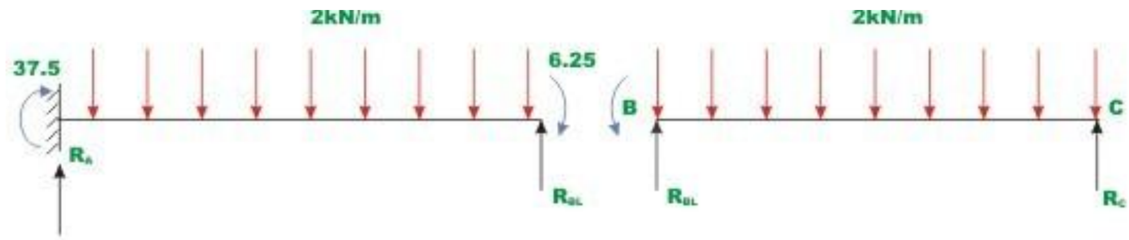
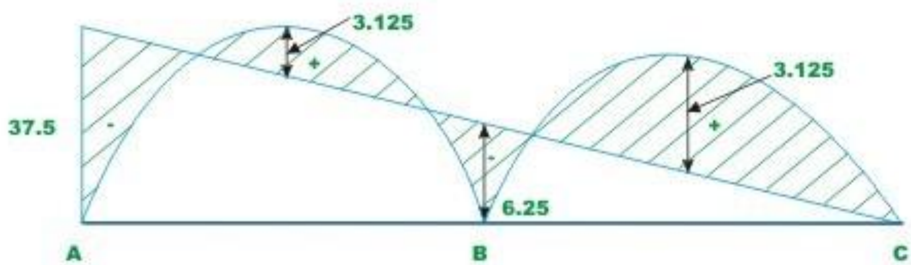


Fig. 12.5 (e) Free - body diagram of two members

S . F . D



B.M.D

Fig. 12.5 (f) Shear force and bending moment diagrams

In span AB , $\sum M_B = 0$

$$R_A \times 10 - 37.5 - 2 \times 10 \times 5 + 6.25 = 0$$

$$R_A = 13.125 \text{ kN} \qquad (\uparrow)$$

$$R_{BL} = 6.875 \text{ kN} \qquad (\uparrow)$$

Similarly from span BC ,

$$R_C = 3.75 \text{ kN} \qquad (\uparrow)$$

$$R_{BR} = 6.25 \text{ kN} \qquad (\uparrow)$$

The shear force and bending moment diagrams are shown in Fig. 12.5f.

Summary

Here the continuous beam with unyielding supports is analyzed by three-moment equations. The three-moment equations are derived for the case of a continuous beam having different moment of inertia in different spans. The three-moment equations also belong to force method of analysis and in this case, redundants are always taken as support moments. Hence, compatibility equations are derived in terms of three support moments. Few problems are solved to illustrate the procedure.

Chapter 5

The Three-Moment Equations-II

Instructional Objectives

After reading this chapter the student will be able to

1. Derive three-moment equations for a continuous beam with yielding supports.
2. Write compatibility equations of a continuous beam in terms of three moments.
3. Compute reactions in statically indeterminate beams using three-moment equations.
4. Analyse continuous beams having different moments of inertia in different spans and undergoing support settlements using three-moment equations.

Introduction

Previously, three-moment equations were developed for continuous beams with unyielding supports. The support may settle by unequal amount during the life time of the structure. Such future unequal settlement induces extra stresses in statically indeterminate beams. Hence, one needs to consider these settlements in the analysis. The three-moment equations developed previously could be easily extended to account for the support yielding. In the next section three-moment equations are derived considering the support settlements. In the end, few problems are solved to illustrate the method.

Derivation of Three-Moment Equation

Consider a two span of a continuous beam loaded as shown in Fig.13.1. Let M_L, M_C and M_R be the support moments at left, center and right supports respectively. As stated in the previous lesson, the moments are taken to be positive when they cause tension at the bottom fibers. I_L and I_R denote moment of inertia of left and right span respectively and l_L and l_R denote left and right spans respectively. Let δ_L, δ_C and δ_R be the support settlements of left, centre and right supports respectively. δ_L, δ_C and δ_R are taken as negative if the settlement is downwards. The tangent to the elastic curve at support C makes an angle θ_{CL} at left support and θ_{CR} at the right support as shown in Fig. 13.1. From the figure it is observed that,

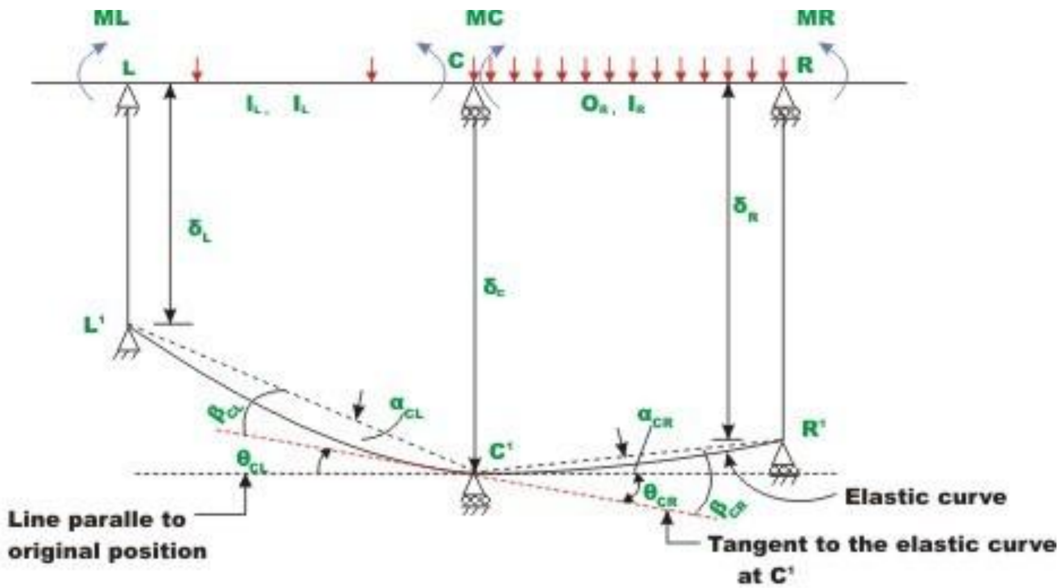


Fig. 13.1 Continuous beam with support settlement

$$\theta_{CL} = \theta_{CR} \tag{13.1}$$

The rotations β_{CL} and β_{CR} due to external loads and support moments are calculated from the M/EI diagram .They are :

$$\beta_{CL} = \frac{A_L \bar{x}_L}{EI_L l_L} + \frac{M_L l_L}{6EI_L} + \frac{M_C l_L}{3EI_L} \tag{13.2a}$$

$$\beta_{CR} = \frac{A_R \bar{x}_R}{EI_R l_R} + \frac{M_R l_R}{6EI_R} + \frac{M_C l_R}{3EI_R} \tag{13.2b}$$

The rotations of the chord $L'C'$ and $C'R'$ from the original position is given by

$$\alpha_{CL} = \frac{\delta_L - \delta_C}{l_L} \tag{13.3a}$$

$$\alpha_{CR} = \frac{\delta_R - \delta_C}{l_R} \tag{13.3b}$$

From Fig. 13.1, one could write,

$$\theta_{CL} = \alpha_{CL} - \beta_{CL} \tag{13.4a}$$

$$\theta_{CR} = \beta_{CR} - \alpha_{CR} \tag{13.4b}$$

Thus, from equations (13.1) and (13.4), one could write,

$$\alpha_{CL} - \beta_{CL} = \beta_{CR} - \alpha_{CR} \tag{13.5}$$

Substituting the values of $\alpha_{CL}, \alpha_{CR}, \beta_{CL}$ and β_{CR} in the above equation,

$$M_L\left(\frac{l_L}{I_L}\right) + 2M_C\left(\frac{l_L}{I_L} + \frac{l_R}{I_R}\right) + M_R\left(\frac{l_R}{I_R}\right) = -\frac{6A_R\bar{x}_R}{I_R l_R} - \frac{6A_L\bar{x}_L}{I_L l_L} + 6E\left(\frac{\delta_L - \delta_C}{l_L}\right) + 6E\left(\frac{\delta_R - \delta_C}{l_R}\right)$$

This may be written as

$$M_L\left(\frac{l_L}{I_L}\right) + 2M_C\left(\frac{l_L}{I_L} + \frac{l_R}{I_R}\right) + M_R\left(\frac{l_R}{I_R}\right) = -\frac{6A_R\bar{x}_R}{I_R l_R} - \frac{6A_L\bar{x}_L}{I_L l_L} - 6E\left[\left(\frac{\delta_C - \delta_L}{l_L}\right) + \left(\frac{\delta_C - \delta_R}{l_R}\right)\right] \tag{13.6}$$

The above equation relates the redundant support moments at three successive spans with the applied loading on the adjacent spans and the support settlements.

Example 1

Draw the bending moment diagram of a continuous beam BC shown in Fig.13.2a by three moment equations. The support B settles by 5mm below A and C. Also evaluate reactions at A, B and C. Assume EI to be constant for all members and $E = 200$ GPa, $I = 8 \times 10^6$ mm^4

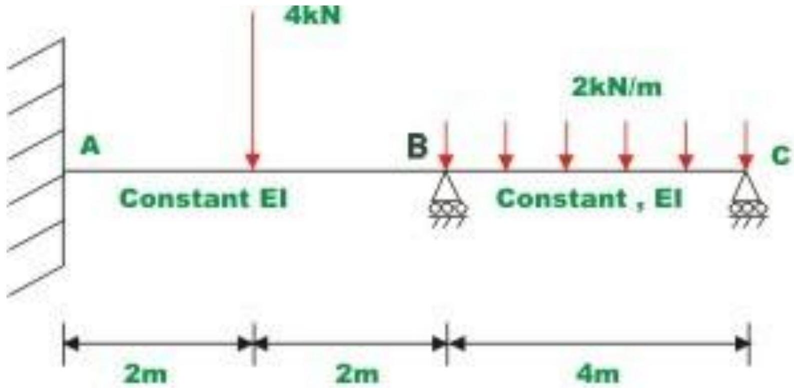


Fig. 13.2(a) Example 13.1

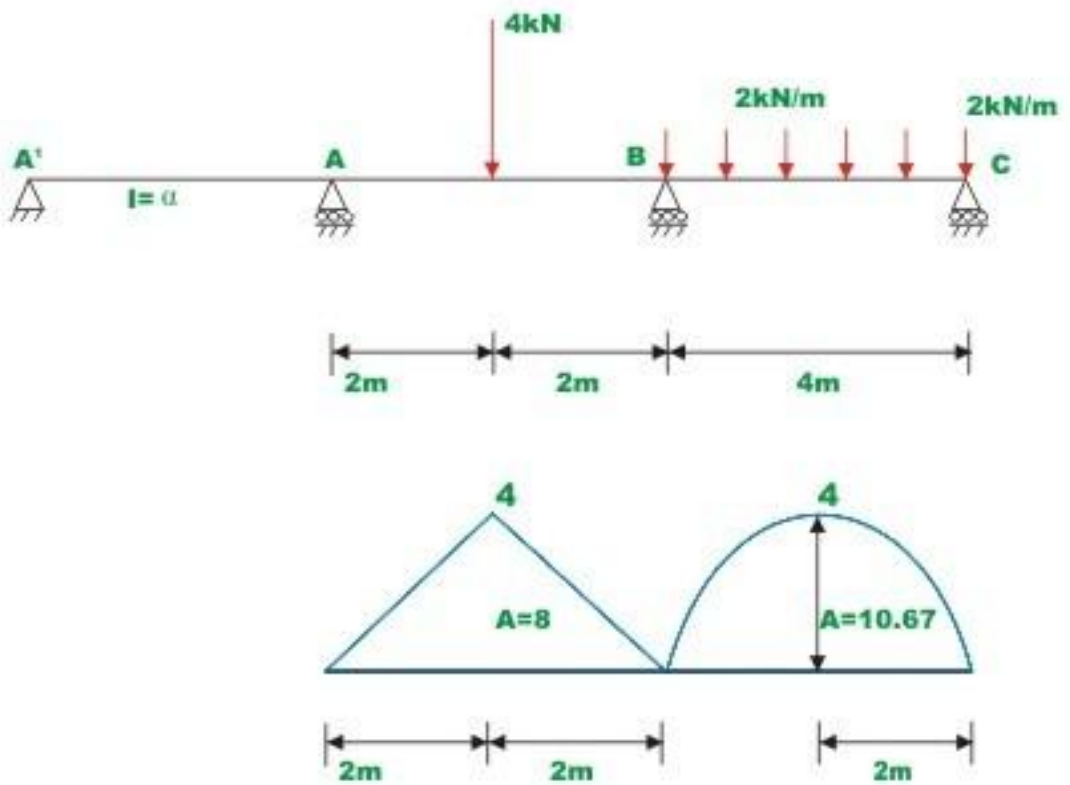


Fig. 13.2(b) Bending moment diagram due to applied loading

Assume an imaginary span having infinitely large moment of inertia and arbitrary span L' left of A as shown in Fig.13.2b . Also it is observed that moment at C is zero.

The given problem is statically indeterminate to the second degree. The moments M_A and M_B,the redundants need to be evaluated. Applying three moment equation to the span *A'AB*,

$$\delta_L = \delta_C = 0 \text{ and } \delta_R = -5\times10^{-3}\,m$$

$$M'_A\left(\frac{L'}{\infty}\right)+2M_A\left(\frac{L'}{\infty}+\frac{4}{I}\right)+M_B\left(\frac{4}{I}\right)=-\frac{6\times8\times2}{I(4)}-6E\left(0+\frac{0-(-5\times10^{-3})}{4}\right)$$

$$8M_A+4M_B=-24-6EI\times\frac{5\times10^{-3}}{4} \qquad (1)$$

Note that, $EI=200\times10^{9}\times\frac{8\times10^{6}\times10^{-12}}{10^{3}}=1.6\times10^{3}\,\text{kNm}^2$

Thus,

$$8M_A+4M_B=-24-6\times1.6\times10^{3}\times\frac{5\times10^{-3}}{4}$$

$$8M_A+4M_B=-36 \qquad (2)$$

Again applying three moment equation to span ABC the other equations is obtained. For this case, $\delta_L=0$, $\delta_C=-5\times10^{-3}\,m$ (negative as the settlement is downwards) and $\delta_R=0$.

$$M_A\frac{4}{I}+2M_B\left(\frac{4}{I}+\frac{4}{I}\right)=-\frac{24}{I}-\frac{6\times10.667\times2}{I\times4}-6E\left(\frac{-5\times10^{-3}}{4}-\frac{5\times10^{-3}}{4}\right)$$

$$4M_A+16M_B=-24-32+6\times1.6\times10^{3}\times\frac{10\times10^{3}}{4}$$

$$4M_A+16M_B=-32 \qquad (3)$$

Solving equations (2) and (3),

$$M_B=-1.0\,\text{kN.m}$$
$$M_A=-4.0\,\text{kN.m} \qquad (4)$$

Now, reactions are calculated from equations of static equilibrium (see Fig.13.2c).

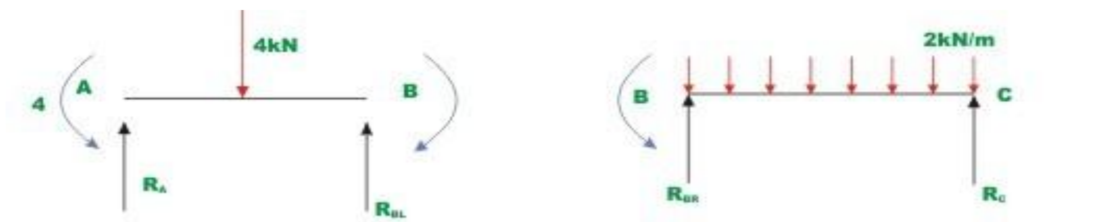
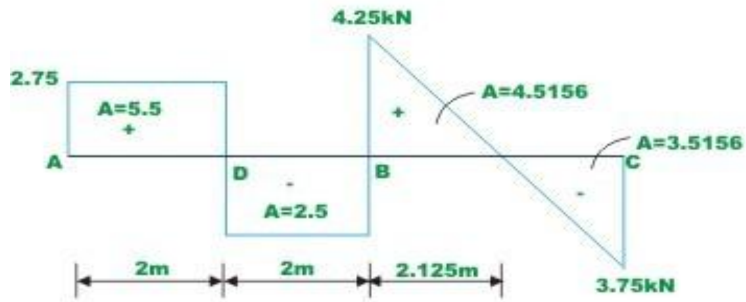
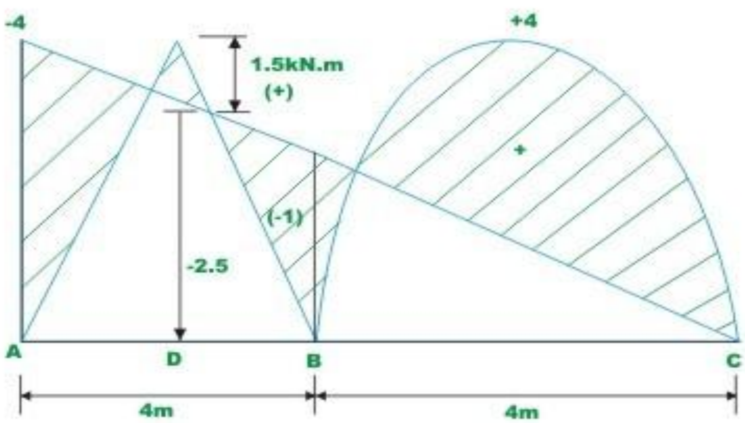


Fig.13.2 (c) Free - body diagram of two members



Shear force diagram



Bending moment diagram

Fig.13.2(d) Shear force and bending moment diagram

Thus,

$$R_A = 2.75 \text{kN} (\uparrow)$$
$$R_{BL} = 1.25 \text{kN} (\uparrow)$$
$$R_{BR} = 4.25 \text{kN} (\uparrow)$$
$$R_C = 3.75 \text{kN} (\uparrow)$$

The reactions at B,

$$R_B = R_{BR} + R_{BL} = 5.5 \text{ kN} \quad (5)$$

The area of each segment of the shear force diagram for the given continuous beam is also indicated in the above diagram. This could be used to verify the previously computed moments. For example, the area of the shear force diagram between A and B is 5.5 kN.m.This must be equal to the change in the bending moment between A and D, which is indeed the case ($-4 - 1.5 = 5.5 \text{ kN.m}$). Thus, moments previously calculated are correct.

Example 2

A continuous beam $ABCD$ is supported on springs at supports B and C as shown in Fig.13.3a. The loading is also shown in the figure. The stiffness of springs is $k_B = \frac{EI}{20}$ and $k_C = \frac{EI}{30}$ .Evaluate support reactions and draw bending moment diagram. Assume EI to be constant.

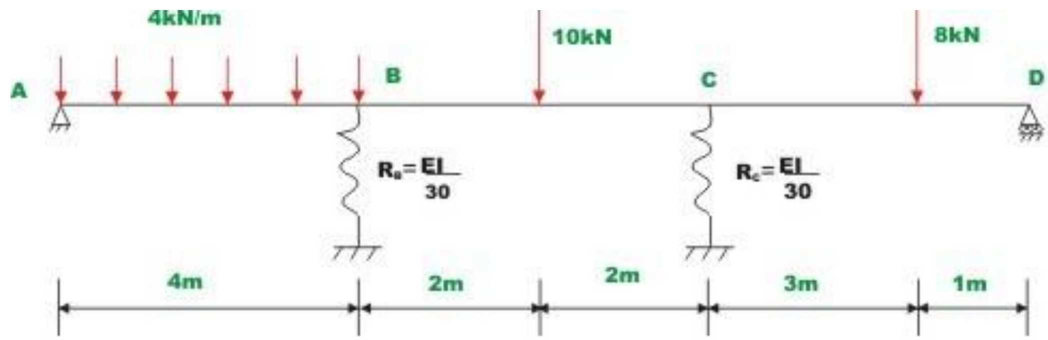


Fig.13.3(a) Continuous beam of Example 13.2

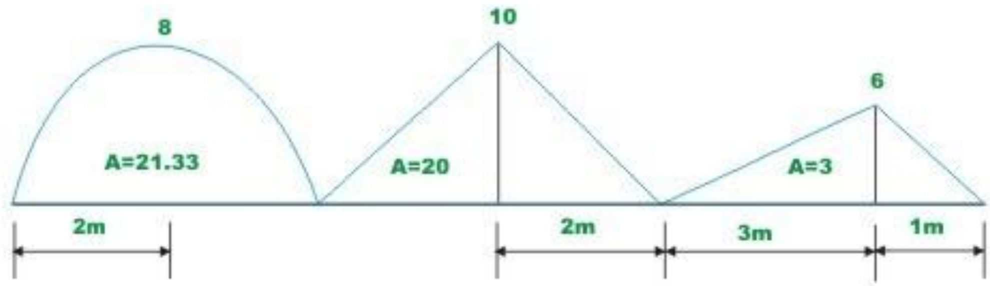


Fig.13.3(b) Bending moment diagram on simple spans due to applied loading

Fig.13.3(c) Computation of reactions

In the given problem it is required to evaluate bending moments at supports B and C. By inspection it is observed that the support moments at A and D are zero. Since the continuous beam is supported on springs at B and C, the support settles. Let R_B and R_C be the reactions at B and C respectively. Then the support settlement at B and C are $\frac{R_B}{k_B}$ and $\frac{R_C}{k_C}$ respectively. Both the settlements are negative and in other words they move downwards. Thus,

$$\delta_A = 0 \text{ , } \delta_B = \frac{-20R_B}{EI} \text{ , } \delta_C = \frac{-30R_C}{EI} \text{ and } \delta_D = 0 \qquad (1)$$

Now applying three moment equations to span ABC (see Fig.13.2b)

$$M_A\left(\frac{4}{I}\right) + 2M_B\left(\frac{4}{I} + \frac{4}{I}\right) + M_C\left(\frac{4}{I}\right) = -\frac{6\times 21.33\times 2}{I\times 4} - \frac{6\times 20\times 2}{I\times 4} - 6E\left\{\frac{20R_B}{4EI} + \frac{-\frac{20R_B}{EI} + \frac{30R_C}{EI}}{4}\right\}$$

Simplifying,

$$16M_B + 4M_C = -124 + 60R_B - 45R_C \qquad (2)$$

Again applying three moment equation to adjacent spans BC and CD ,

$$M_B\left(\frac{4}{I}\right) + 2M_C\left(\frac{4}{I} + \frac{4}{I}\right) = -\frac{60}{I} - \frac{(6\times 9\times 2 + 6\times 3\times \frac{2}{3}\times 1)}{I\times 4} - 6E\left\{\frac{-\frac{30R_C}{EI} + \frac{20R_B}{EI}}{4} + \frac{-30R_C}{4EI}\right\}$$

$$4M_B + 16M_C = -90 + 90R_C - 30R_B \qquad (3)$$

In equation (2) and (3) express R_B and R_C in terms of M_B and M_C (see Fig.13.2c)

$$\begin{aligned}
R_A &= 8 + 0.25M_B \ (\uparrow) \\
R_{BL} &= 8 - 0.25M_B \ (\uparrow) \\
R_{BR} &= 5 + 0.25M_C - 0.25M_B \ (\uparrow) \\
R_{CL} &= 5 + 0.25M_B - 0.25M_C \ (\uparrow) \\
R_{CR} &= 2 - 0.25M_C \ (\uparrow) \\
R_D &= 6 + 0.25M_C \ (\uparrow)
\end{aligned} \qquad (4)$$

Note that initially all reactions are assumed to act in the positive direction (i.e. upwards) .Now,

$$R_B = R_{BL} + R_{BR} = 13 - 0.5M_B + 0.25M_C$$

$$R_C = R_{CL} + R_{CR} = 7 + 0.25M_B - 0.5M_C \qquad (5)$$

Now substituting the values of R_B and R_C in equations (2) and (3),

$$16M_B + 4M_C = -124 + 60(13 - 0.5M_B + 0.25M_C) - 45(7 + 0.25M_B - 0.5M_C)$$

Or,

$$57.25M_B - 33.5M_C = 341 \quad (6)$$

And from equation 3,

$$4M_B + 16M_C = -90 + 90\left(7 + 0.25M_B - 0.5M_C\right) - 30\left(13 - 0.5M_B + 0.25M_C\right)$$

Simplifying,

$$-33.5M_B + 68.5M_C = 150 \quad (7)$$

Solving equations (6) and (7)

$$\begin{aligned} M_C &= 7.147 \text{ kN.m} \\ M_B &= 10.138 \text{ kN.m} \end{aligned} \quad (8)$$

Substituting the values of M_B and M_C in (4),reactions are obtained.

$$R_A = 10.535 \text{ kN} \quad (\uparrow) \qquad R_{BL} = 5.465 \text{ kN} \quad (\uparrow)$$

$$R_{BR} = 4.252 \text{ kN} \quad (\uparrow) \qquad R_{CL} = 5.748 \text{ kN} \quad (\uparrow)$$

$$R_{CR} = 0.213 \text{ kN} \quad (\uparrow) \qquad R_D = 7.787 \text{ kN} \quad (\uparrow)$$

$$R_B = 9.717 \text{ kN} \quad (\uparrow) \quad \textit{and } R_C = 5.961 \text{ kN} \quad (\uparrow)$$

The shear force and bending moment diagram are shown in Fig. 13.2d.

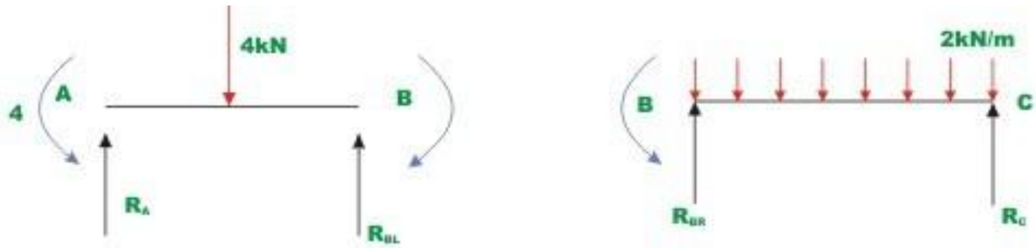
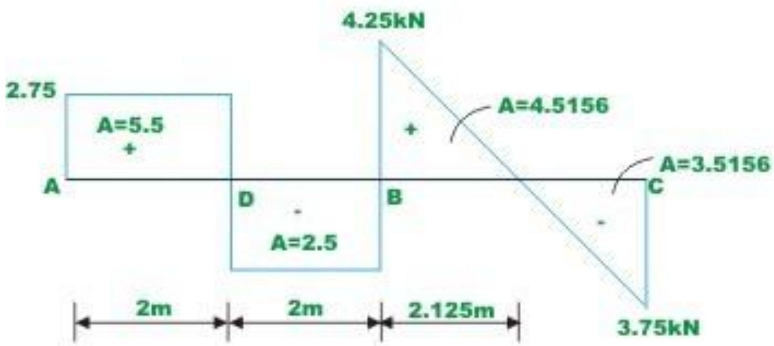
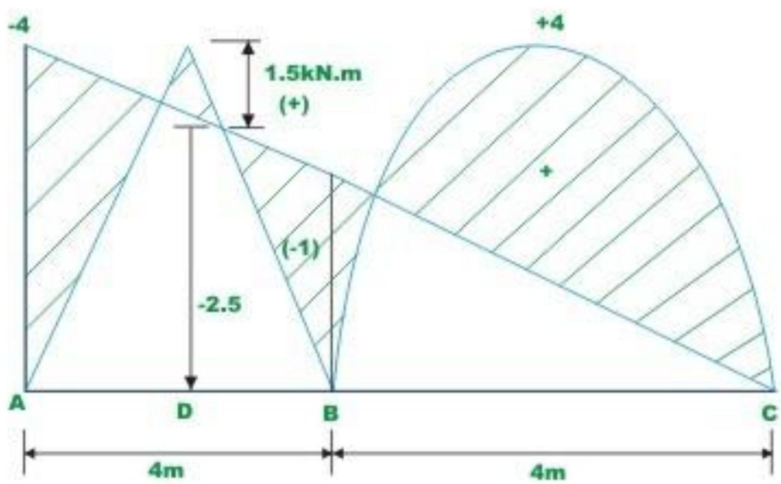


Fig.13.2 (c) Free - body diagram of two members



Shear force diagram



Bending moment diagram

Fig.13.2(d) Shear force and bending moment diagram

Example 3

Sketch the deflected shape of the continuous beam *ABC* of example 1.
The redundant moments M_A and M_B for this problem have already been computed in Example 1 above.They are,

$$M_B = -1.0\,\text{kN.m}$$
$$M_A = -4.0\,\text{kN.m}$$

The computed reactions are also shown in Fig.13.2c.Now to sketch the deformed shape of the beam it is required to compute rotations at *B* and *C*. These joints rotations are computed from equations (13.2) and (13.3).
For calculating θ_A , consider span *A'AB*

$$\theta_A = \beta_{AR} - \alpha_{AR}$$

$$= \frac{A_R \bar{x}_R}{EI_R l_R} + \frac{M_B l_R}{6EI_R} + \frac{M_A l_R}{3EI_R} - \left(\frac{\delta_B - \delta_A}{4}\right)$$

$$= \frac{6\times 8\times 2}{1.6\times 10^3\times 4} + \frac{M_B\times 4}{1.6\times 10^3\times 6} + \frac{M_A\times 4}{1.6\times 10^3\times 3} - \left(\frac{\delta_B - \delta_A}{4}\right)$$

$$= \frac{6\times 8\times 2}{1.6\times 10^3\times 4} + \frac{(-1)\times 4}{1.6\times 10^3\times 6} + \frac{(-4)\times 4}{1.6\times 10^3\times 3} + \left(\frac{5\times 10^{-3}}{4}\right)$$

$$= 0 \tag{1}$$

For calculating θ_{BL} , consider span *ABC*

$$\theta_{BL} = \alpha_{BL} - \beta_{BL}$$

$$= -\left(\frac{A_L \bar{x}_L}{EI_L l_L} + \frac{M_A l_L}{6EI_L} + \frac{M_B l_L}{3EI_L}\right) + \left(\frac{\delta_A - \delta_B}{l_L}\right)$$

$$= -\left(\frac{8\times 2}{1.6\times 10^3\times 4} + \frac{(-4)\times 4}{1.6\times 10^3\times 6} + \frac{(-1)\times 4}{1.6\times 10^3\times 3}\right) + \left(\frac{5\times 10^3}{4}\right)$$

$$= 1.25\times 10^{-3}\ \text{radians} \tag{2}$$

For θ_{BR} consider span ABC

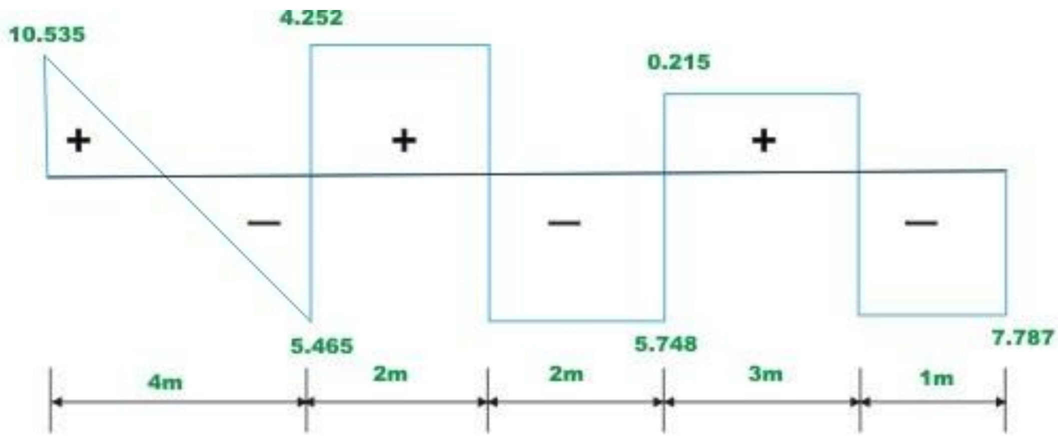
$$\theta_{BR} = \left[\frac{10.67\times 2}{1.6\times 10^3\times 4} + \frac{(-1)\times 4}{1.6\times 10^3\times 3}\right] - \left[0 + \frac{5\times 10^3}{4}\right]$$

$$= -1.25\times 10^{-3} \text{ radians} \qquad (3)$$

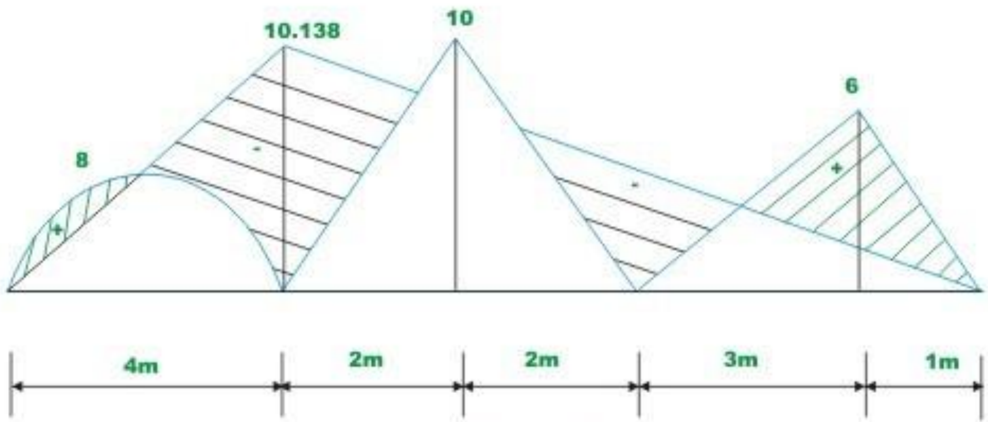
$$\theta_C = -\left[\frac{10.67\times 2}{1.6\times 10^3\times 4} + \frac{(-1)\times 4}{1.6\times 10^3\times 3}\right] - \left[\frac{\delta_B - \delta_C}{4}\right]$$

$$= -3.75\times 10^{-3} \text{ radians.} \qquad (4)$$

The deflected shape of the beam is shown in Fig. 13.4.



Shear force diagram



Bending moment diagram
Fig.13.3(d)

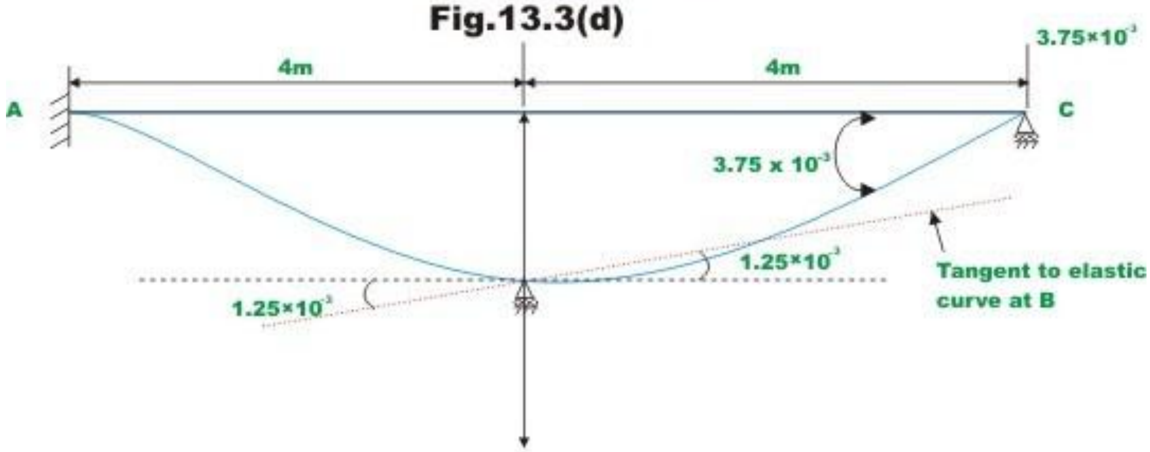


Fig.13.4(a) Elastic curve Example 13.3

Summary

Earlier ,the continuous beams with unyielding supports are analysed using three-moment equations. Here, the three-moment-equations developed in the previous lesson are extended to account for the support settlements. The three-moment equations are derived for the case of a continuous beam having different moment of inertia in different spans. Few examples are solved to illustrate the procedure of analysing continuous beams undergoing support settlements using three-moment equations.